Students' formalising process of the limit concept

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The concept of limit is the foundation for many concepts such as the derivative and the integral in advanced mathematics (Parameswaran, 2007; Tall, 1992). According to the literature, there are two different kinds of notions of the limit, a dynamic limit notation and a formal limit notion. The dynamic limit notion is considered as

$$\text{as } x \to a \Rightarrow f(x) \to L$$

and read, "as x approaches a, $f(x)$ approaches L". The formal limit definition is given as follows: If the condition,

$$\text{For any } \varepsilon > 0, \text{ there exist } \delta > 0 \text{ such that } |\, x - a \,| < \delta \Rightarrow |\, f(x) - L \,| < \varepsilon$$

is satisfied, the limit of the function of f at the point of a is L.

The limit concept has been a research topic in mathematics education for years and in the literature it is a broadly accepted fact that the limit is a difficult notion for most students (e.g., Cornu, 1991; Cottrill et al., 1996; Mamona-Downs, 2001; Przenioslo, 2004; Roh, 2007; Sierpinska, 1987; Tall, 1993; Tall & Vinner, 1981; Williams, 1991). In the light of the relevant literature, it is indicated that understanding the formal limit notion is difficult for most students (e.g., Cornu, 1991; Swinyard, 2011). Some researchers (e.g., Tall & Vinner, 1981; Williams, 1991) have argued that the dynamic limit notion is easy and natural for the students, however, it prevents the students from understanding the formal limit notion. Swinyard and Locwood (2007) have stated that students must first consider a range of output values around the candidate for the limit and then project these back to the domain. Swinyard and Locwood, describe this as to reverse one's thinking and argue this is the main difficulty in constructing the formal limit notion.

As Swinyard (2011) has emphasised, there is little research about students' understanding of the formal limit notion, despite the fact that understanding of the formal limit notion is difficult for most of the students. Additionally, he

has indicated that little is known about how students reason coherently about the formal definition of limit. Swinyard (2011) notes there need to be more studies into the formal understanding of the limit concept. The study reported here aimed to investigate students' understanding of the formal limit notion by an instructional treatment designed in the light of the relevant literature.

Relevant literature

Some researchers have investigated the students' concept images and definitions of the limit (e.g., Mamona-Downs, 2001; Parameswaran, 2007; Przenioslo, 2004; Tall & Vinner, 1981). In this regard, Tall and Vinner call the total cognitive structure, which includes all mental pictures, associated properties and processes, the *concept image*. Vinner (1991) has interpreted the concept image and concept definition as two different 'cells' in the cognitive structure. He has claimed that the interplay between the concept image and concept definition refers to the long-term processes of the concept formation. In recent years, Bingolbali and Monaghan (2008) have revisited the concept image via emphasising the roles of teacher's teaching practices and institution she/he teaches in. They have concluded that the concept images developed by the students are affected by teaching practices and departmental affiliations.

Many researchers have argued that the students have an intuitive understanding about the limit concept (e.g., Tall & Vinner, 1981; Williams, 1991). They concluded that most students develop a concept image including the dynamic limit notion. Moreover, these researchers have claimed that the dynamic limit notion includes the feeling of motion and the students are able to interpret this notion intuitively and easily. On the other hand, according to these researchers, it is not easy to formalise the students' intuitive understanding of the limit notion (e.g., Szydlik, 2000; Williams, 1991). They argue that the dynamic limit notion prevents the students from formalising the limit notion.

Research by Cottrill et al. (1996) focuses on understanding the limit concept based on APOS theory in which a learners' specific mental constructions are called Action, Process, Object and Schema. According to Cottrill et al., the schema of two coordinated processes (as $x \to a$, $f(x) \to L$) is reconstructed in terms of the intervals and inequalities to obtain a process described as "$0 < | x - \alpha | < \delta$ implies $| f(x) - L | < \varepsilon$". Cottrill et al. have concluded that the dynamic limit notion does not prevent the students from understanding the formal limit notion, on the contrary, it is necessary to construct the formal limit notion. In this respect, Swinyard and Lockwood (2007) have argued that the study of Cottrill et al. is particularly a useful starting point, but the formalisation process is not as straightforward as they obtained.

Pinto and Tall (1999, 2002) have argued that the students' construction route for the limit concept depends on their ways of thinking. They have

indicated that the students have two different kinds of mathematical minds, which they name formal thinkers and natural thinkers. These two kinds of thinkers' learning strategies are given in Table 1.

Table 1. Pinto and Tall's different kinds of mathematical minds.

Mathematical minds	Learning strategy
formal thinkers	extracting meaning: beginning with the formal definition and constructing properties by a logical deduction
natural thinkers	giving meaning: to the definition and resulting the theory by building from the earlier concept images

In general, the limit concept is introduced by dynamic limit notion in the high school level informally as is the case in the country where the study was conducted. In the senior secondary *Australian Curriculum: Mathematics*, the concept of limit is introduced informally and then the definition of derivative using limit formula is given. Formal limit definition is given in the university level.

Barbé et al. (2005) argued that the problem of teaching 'limits of functions' in secondary school constitutes a specific mathematical organisation which includes two local mathematical organisations, algebra of limits and topology of limits. The algebra of limits is about the calculation of the limit value. The other mathematical organisation, topology of limits, concerns the existence of the limit of the function. According to Barbé et al., if the mathematical organisations are not linked to each other by the teacher, there will be some problems in the didactic process. Similarly, Mamona-Downs (2001) emphasised the necessity of explaining the problem of the clash between the intuitive image including the dynamic approaching and the static image evoked by the definition of limit concept.

Methodology

This is a qualitatively designed study and the research design is a teaching experiment. An academic semester teaching experiment was conducted in the course of Analysis I in the Mathematics Education Program in an Education Faculty in order to investigate students' formalising process of the limit concept. The instructor of the course was the researcher. Kelly and Lesh (2000) state that teaching experiment is perhaps the most convenient research type that clearly illustrates distinctive characteristics of research in mathematics and science education. According to Cobb and Steffe (1983), a teaching experiment is an interaction between a group of a students and a researcher, who is also the instructor.

The participants, instruments and research process of the study

The participants of the study were selected from the 28 students taking the one semester class covering the content of limit, continuity, derivative, and integral concepts for the single variable functions. Seven participants were selected on the basis of an open-ended test, developed by the researcher, using the purposive sampling technique (Fraenkel & Allen, 1996). The criteria for selection were the students' definition of the limit concept, their arguments in the algebraic, graphical and formal tasks, and the words they associated with the limit notion.

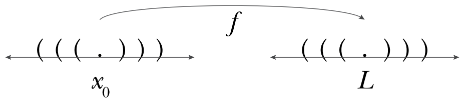
Data for the study were gathered using two clinical interviews. The first interviews were conducted after the concept of derivative had been taught, and the second interview after the concept of integral had been taught. In both interviews, firstly, the limit concept was the focus and then limit notion related to the derivative or integral concepts, which are special limit cases, respectively. The clinical interviews (Clement, 2000) were recorded using a camera. The researcher asked the interview questions one at a time and the students were required to write their calculations and symbolic expressions on the worksheets provided by the researcher. Transcriptions of the recordings were coded by the researcher, in order to detect the key events. Worksheets were also used in the analysis. The students' responses in the test were analysed qualitatively using the content analysis technique (Miles & Huberman, 1994).

Teaching experiment

The instructional treatment in this study was designed in the light of the relevant literature. Formal limit notion was intended to construct on dynamic limit notion step by step as shown in Table 2 in which the notions instructed in class and the meanings were given in order. The students were encouraged to formalise the dynamic limit notion and associate the typology of limit, in which they validated the limit, with the algebra of limit in class discussions. After teaching of the limit concept in this way, the concepts of continuity, derivative, and integral were taught throughout the course.

As Tall (1991) has indicated any mathematical analysis of the notion of derivative must be preceded by the discussion of the limit notion. Hence, the limit case in the concept of derivative was discussed. At first, the rate of change notion of derivative was handled heuristically with a velocity problem. The instantaneous speed of a car was discussed. It was emphasised that the average speed converges to the instantaneous speed while it is approaching to the constant time. Moreover, these discussions were not only conducted in terms of the dynamic limit notion, but also in terms of the formal limit notion. After the introduction of the derivative notion in the speed context, it was also introduced in the context of the slope of sliding secants geometrically (Thompson, 1994). In a similar way, the notion of integral was taught by emphasising the limit case in this concept.

Table 2. Instructional treatment of limit concept.

Notion	Meaning
Convergence notion	$x \to x_0$; taking x values closer and closer to x_0.
Difference between the meanings of convergence in limit concept and in daily life.	In daily life meaning, something is moving. In limit concept, the points we take are changing. There is only feeling of motion based on literature.
Dynamic limit notion as a coordination of two convergence processes $(x \to x_0 \Rightarrow f(x) \to L)$.	Dynamic limit notion : "as $x \to x_0 \Rightarrow f(x) \to L$" or "as x approaches x_0, $f(x)$ approaches L".
Relation between the neighbourhoods of a point and convergence to that point.	Converging xs to x_0 are in the narrower neighbourhoods of x_0. While we are taking closer points, we take narrower neighbourhoods.
Review of coordination of convergences in limit notion with neighbourhoods.	$x \to x_0 \Rightarrow f(x) \to L$ 
Representation of a neighbourhood with inequality.	$N_\delta(x_0) = \{x \in \mathbb{R} \mid \lvert x - x_0 \rvert < \delta\}$ $x_0 - \delta$ $\quad x_0 \quad$ $x_0 - \delta$
Formal limit notion	If the condition, For any $\varepsilon > 0$, there exists $\delta > 0$ such that $\lvert x - a \rvert < \delta \Rightarrow \lvert f(x) - L \rvert < \varepsilon$ is satisfied, the limit of the function of f at the point of a is L.

Results

The results of the study have shown that participants' formalisation processes can be categorised into two groups. Four of the participants will be named as the dynamic participants whereas the other participants will be named as the formal participants.

Table 3. The participants.

Participants	Formalisation approach
Dynamic participants: D1, D2, D3, D4.	Formalising concept images including dynamic limit notion.
Formal participants: F1, F2, F3.	Deducing meaning from the formal limit definition.

Dynamic limit notion

It was seen that dynamic participants had the dynamic limit images and they formalised their concept images including dynamic limit notion. When the limit notion was the focus of questions in the interviews, the dynamic participants provided a descriptive definition including the dynamic limit

notion. In addition, they reflected the dynamic limit notion and the feeling of motion in their expressions and arguments. It was seen that the words like 'approaching points', 'approach' were salient in their expressions and arguments in the interviews. Dynamic participants related the feeling of motion in their dynamic images to the neighbourhoods in formalising process as in the instructional treatment.

The first dynamic participant, D1

In this study, D1 was the most successful dynamic participant about formalising the dynamic limit notion. D1 showed that she began to relate her strong dynamic image to the formal definition by relating the notions of convergence and neighbourhood from the outset of the study. She reflected the feeling of motion not only by her expressions but also by her hand movements while describing the dynamic limit.

> D1: I can say that while the points are approaching to a value [she was moving her hands as narrower neighbourhoods] the values of $f(x)$ that were taken as corresponding points are approaching only one point, this point is our limit value.

In this formalisation process, she reflected the dynamic processes coordinated via the function and related these processes with the narrower neighbourhoods.

> D1: I can take small neighbourhoods arbitrarily. Actually, the points are not moving. I am changing the neighbourhoods.

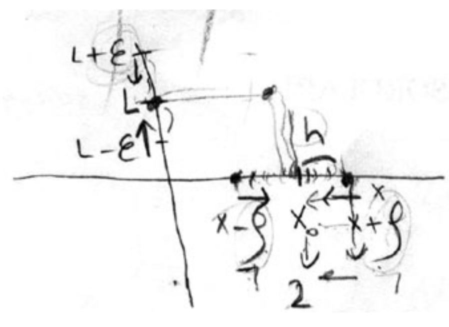


Figure 1. Figure drawn by D1 during the interview.

In the second interview, D1 made progress in the formalisation process by relating the inequalities in formal limit notion with the narrower neighbourhoods.

On the other hand, D1 used the formal limit notion routinely in order to prove the limit of a function in the first interview. To prove a limit, ε is chosen arbitrarily and existence of δ, dependent on ε, is proved such that "$| x - x_0 | < \delta \Rightarrow | f(x) - L | < \varepsilon$". It was apparent that D1 proved the limit given by rote learning because she explained the relationship between ε and δ with

direction of dynamic limit process from domain to range in another session of the first interview. In the second interview, D1 seemed to give up the direction of the dynamic limit process in order to explain the relationship between ε and δ. She said that she should have found δ dependent on ε, but she did not know why. That is, D1 had dynamic limit image and made progress in the formalisation process by relating dynamic processes with the narrower neighbourhoods and inequalities. However, she demonstrated that she could not understand the quantifiers in formal limit notion since the direction of her feeling of motion was from domain to range.

The second dynamic participant, D2

D2, who was a less successful dynamic participant than D1, had a weaker dynamic limit image than D1 in the first interview. It was seen that she coordinated the convergences in domain and range via the function; however, she had difficulties in explaining this. D2 frequently needed to use the graphical representation, while D1 was able to reflect her concept image via her explanations and hand movements.

Despite the fact that D2 was not able to reflect the relation between the notions of convergence and neighbourhood as clear as D1 did, she focused on taking the image of neighbourhoods and reflected the feeling of motion with approaching points in the neighbourhoods as in the following:

I: Okay, what is the relation between the convergence and the neighbourhood you mention?

D2: That is, we are taking the approximating points in the neighbourhood. For instance, when I think δ as 2, I consider two units of neighbourhoods and I begin to investigate the neighbourhood with two units [she was explaining it on the coordinate plane and she was narrowing the neighbourhoods down].

That is, D2 began to develop the relation between the notions of convergence and neighbourhood, but she could not relate the convergence notion with inequalities in formal limit notion. She did not use formal limit definition even routinely.

The third dynamic participant, D3

The third dynamic participant (D3) also reflected use of a dynamic limit image. In the first interview, D3 considered the neighbourhoods as static, even though she attempted to use the notion of neighbourhood in her explanations. D3 made progress in the formalisation process until the second interview. It is seen that she began to consider the notion of convergence with the narrower neighbourhoods and explained the relationship between ε and δ with direction of dynamic limit process that is from domain to range in the second interview.

I: What does the convergence mean?

D3: The convergence means gradually narrower intervals [she draws a coordinate system and shows on it]. For instance, let this point be $x_0 + 3$, then this point is closer, I am making these small, let $x_0 + 1$, that is I am gradually approaching to this point, that is I am making the interval smaller between these points.

…

D3: This interval. Now, epsilon neighbourhood of $f(x_0)$ is here. We took delta arbitrary. Epsilon changes depending on delta.

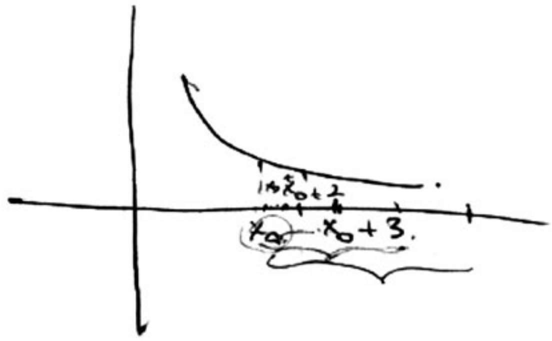


Figure 2. Figure drawn by D3.

That is, D3's progress from the first interview to the second improved in a great extent. Although D3's formalisation was weak in the first interview, she reached the same formalisation as D2's in the second interview. She had the feeling of motion in narrower neighbourhoods while she considered the convergence as close points in the neighbourhood in the first interview.

The fourth dynamic participant, D4

The last dynamic participant (D4) demonstrated that she gained the convergences in the domain and range and coordinated them via the function. It was seen that D4 grasped the relation between convergence and neighbourhoods, but her formalisation was weaker than D3. Even she could not clearly demonstrate her dynamic image with her explanations, her hands' movement showed her dynamic image since she was approaching her fingers like the narrower neighbourhoods in the first interview.

D4: The values it takes while approaching, indicate the limit to us. From left and right, while we are approaching to left and right neighbourhoods [she was moving her fingers like narrower neighbourhoods in the air] while we approach, also the case of the approaching images.

I: Okay, what do the approaching points mean?

D4: That is each value we took is close to x_0, the values which go to the closest point approach, how can I say [pause] the difference is the smallest between the point and x_0. We are taking the neighbourhood arbitrarily; we cannot take specially, because there are infinitely many numbers.

On the other hand, D4's notion of neighbourhood was distorted. She considered the neighbourhood as the end points of the interval.

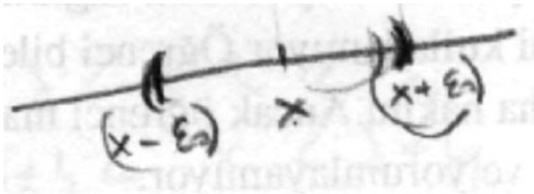


Figure 3. D4's neighbourhood figure.

D4's notions of convergence and neighbourhood were much clearer in the second interview. She had the feeling of motion with neighbourhoods, however, her notion of neighbourhood was still distorted.

Formal limit notion

It was seen that formal participants' formalisation processes of the limit concept were different from the dynamic participants' processes. Formal participants deduced meaning from the formal definition. At first they extracted the meaning from inequalities in the formal definition. At this level, formal participants regarded the limit notion as static, because x and $f(x)$ values did not move. It was seen that formal participants also developed a concept image during the formalisation process of the concept. In formalising process, they related the notions of dynamic and formal limit. They gained feeling of motion in points in fixed neighbourhood at first. Then they changed the neighbourhoods as approaching.

The first formal participant, F1

F1 was the most successful formal participants. The formal participants gave the formal or distorted formal definitions. The successful formal participants expressed the connectives of the formal definitions in their native language. The definition of F1 is shown as following:

> For an arbitrary $\varepsilon > 0$, $\exists\delta > 0$ should be exist such that for $\forall x \in \mathbb{R}$, which satisfy $| x - x_0 | < \delta$, $| f(x) - L | < \varepsilon$ should be satisfied.

Formal participants demonstrated that they also had the dynamic limit notion and they attempted to formalise the concept by relating the meaning they extracted from the formal definition to the dynamic limit notion. At the beginning of their formalisation process, the formal participants perceived the notion of neighbourhood as static close points. In other words, contrary to the dynamic participants, they did not have a dynamic image involving feeling of motion. In the following part, a quotation from the first interview with F1 is presented:

F1: The limit, L is here (he was explaining on the coordinate axis). ε-neighbourhood is here. Images of the points in δ interval will be in ε-neighbourhood.

The following explanations of F1 from the first interview refer to his extracting meaning from the formal definition.

F1: It means that $f(x)$ minus L is less than epsilon… that is $f(x)$ minus L is less than epsilon means you are taking an image and when I subtract it from here, it should be smaller than this interval [he was indicating epsilon interval]. From this part [he was indicating $|f(x) - L| < \varepsilon$]. That is, for the images we took, the subtraction from images of x minus the limit should be less than epsilon. That is, xs should be in the delta neighbourhood.

Moreover, F1 generally emphasised the direction in proving process of a limit from range to domain.

I: What does this part of your definition mean? [The interviewer was indicating the part of "For arbitrary $\varepsilon > 0$, there exist at least one $\delta > 0$" in his definition.]

F1: I have already known that this limit exists. I am taking an interval here (indicating y-axis). This is arbitrary. The values are here [indicating the interval on y-axis]. Let's say $f(x_1)$, then x_1 should be in δ interval.

In the second interview, it was seen that F1 developed his dynamic limit notion in his image. In this interview, it was also seen that he gained the feeling of motion in the points, while he did not have this feeling in the first interview. Even though he started to relate the formal definition and the dynamic image he developed, he still considered the neighbourhoods as static:

F1: Again take a graph [he was drawing graph of an increasing function]. We want limit of this at x_0. Let there exist a delta neighbourhood here. When x values approach to x_0, then images approach to $f(x_0)$ in epsilon neighbourhood.

I: Okay, you drew an arrow here, you said "approach". What was the convergence?

F1: Convergence means taking closer points to x_0.

The second formal participant, F2

The formalisation process of F2 was similar to F1's process. F2 gave the formal definition in which she expressed the connectives in her native language as F1 did. It is seen that F2 considered the notion of convergence as the closest points:

I: What does it mean to approach?

F2: It means going the closest point.

I: How do they go?

F2: For instance, in order to approach from right, the numbers were approaching by decreasing.

Despite the fact that F2 began to develop the dynamic limit image in the first interview, it is seen that she did not coordinate the convergences via the function. In the first interview, F2 described the dynamic limit notion as only one convergence process rather than the coordination of two processes. In the second interview, it was seen that F2 coordinated the convergences. Moreover, it was seen that she thought the notion of convergences as taking closer points. That is, she had the feeling of motion in the points rather than in the neighbourhoods. In other words, she thought the neighbourhoods as static:

F2: These values that are in the neighbourhood become closer than they are in the previous one.

I: You said there was not a motion like in our daily life. So, how do they become closer than the previous one?

F2: I am taking much closer points in the neighbourhood. Neighbourhood is fixed.

The third formal participant, F3

The last formal participants, F3 was less successful than F1 and F2. In the first interview, F3 used the formal definition routinely in order to prove the given limit and he constantly attempted to base his ideas on the definition:

F3: xs are going to x_0, yes. So, do we take δ for x? … These are inferred from this formula.

When the definition of the limit concept was asked in the second interview, it was seen that he began to coordinate the convergences in the domain and range:

I: Could you give the definition of the limit concept?

F3: While xs go a fixed point, their images go somewhere else.

In the second interview, F3 was also able to explain the convergence notion by taking the images of some specific points he thought as close points:

F3: Approaching means that the points are getting close to x_0. For instance, if we say 2 for x_0, let's start with 0 for instance. Let's say 0, 0.1, 0.2, 0.3, 0.4, 0.5 … they are going to 2.

I: You said "going". Are these points moving?

F3: No, I am thinking that they are going.

Participants' construction of the limit situation in derivative

Dynamic and formal participants constructed the limit situation in different ways also for the notion of derivative. The dynamic participants made instantaneous speed interpretation of derivative, while the formal participants interpreted the limit situation in the derivative notion geometrically.

Dynamic participants reflected their dynamic images with their instantaneous speed interpretation. For instance, D1 was able to make instantaneous speed interpretation of derivative with the intervals, which narrow down.

D1: I can find the average speed in time period by calculating change of location. Then, I make smaller time intervals [she was moving her hands like narrower neighbourhoods] in order to obtain the instantaneous speed. By taking smaller time intervals, the values that I take decrease or increase. I can estimate the instantaneous speed.

D2 also recalled the instantaneous speed interpretation correctly as did D1. Moreover, D3 was able to make instantaneous speed interpretation of derivative in spite of the fact that she was not as successful as D1 and D2 were.

D3: We cannot find instantaneous speed but we can approach the instantaneous speed such that we find it in this way. For instance, when we would like to find the velocity at the third second, we approach the third second. For instance, we find at 2.9 second, at 3.1 second… Let's think about the neighbourhood of 3. We are taking the seconds in this neighbourhood to find the close second to 3.

The last dynamic participant, D4, recalled some memorised knowledge in respect to the derivative notion, however, it was seen that she did not gain the limit cases in this notion.

With regard to the formal participants' interpretation for the concept of derivative, the most successful formal participant F1's expressions and graph of was as following.

F1: I want to find tangent of this at a point. I indicate another point let say x_1. I draw the chord. Now…
[pause, writing $\lim_{x_1 \to x_0} \dfrac{f(x_1)-f(x_0)}{x_1 - x_0}$]

I was explaining the slope at that point. The more x_1s approach to x_0, the more I approach the slope of tangent. Here, the more this chord goes

to the right, the more it approaches, that is the more x_1s approach to x_0 [he was approaching the chords towards the tangent].

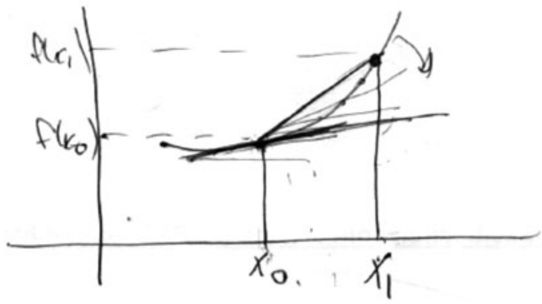


Figure 4. F1's geometric interpretation of the derivative concept.

Even the second formal participant, F2 was not as successful as F1 in the derivative notion, she attempted to interpret the derivative notion geometrically, too. The least successful formal participant, F3 recalled the rules of taking derivative and mathematical expression of limit for the derivative notion as shown in the followings:

F3: Derivative is taking the power down. For me, there is not any other definition. For x^2, $2x$. It is okay, derivative is this.

With regard to the participants' interpretations about the concept of integral, it is seen that neither dynamic participants nor formal participants were successful as they were about the concept of derivative. They attempted to interpret this concept, as it was the convergence of the total sum of rectangle areas to the investigated area.

Discussion and conclusion

In the current study, it is concluded that the students' formalisation process of the limit concept can differ from each other. The findings of this study confirm the findings of Pinto and Tall (1999) in that there are two kinds of construction routes of the limit concept. In general, the findings of this study about the dynamic and formal participants' formalisation ways of the limit concept are compatible with their findings about natural and formal students' ways of thinking, as shown in Table 1, respectively.

On the one hand, there were two different kinds of formalisation process of the limit concept, which are dynamic and formal participants' formalisation routes; on the other hand, in both kinds of routes, the limit concept was formalised by relating the formal definition and the concept image in this study. That is, both of these two kinds of formalisation routes, given in Figure 5 and Figure 6, confirm the Vinner's (1991) claim in that the long-term processes of concept formation include the interplay between the concept

image and concept definition. Therefore, the present study indicates that the dynamic limit notion is essential for both the dynamic and formal students in order to formalise the limit concept. Moreover, it was seen that instructional treatment encouraged the participants to formalise the limit notion in this study. As a result, it was concluded that limit notion may be formalised by relating dynamic limit notion with formal limit definition step by step as in the instructional treatment in this study in university level, if dynamic limit notion is in secondary, formal limit notion is in university level as in Australian curriculum.

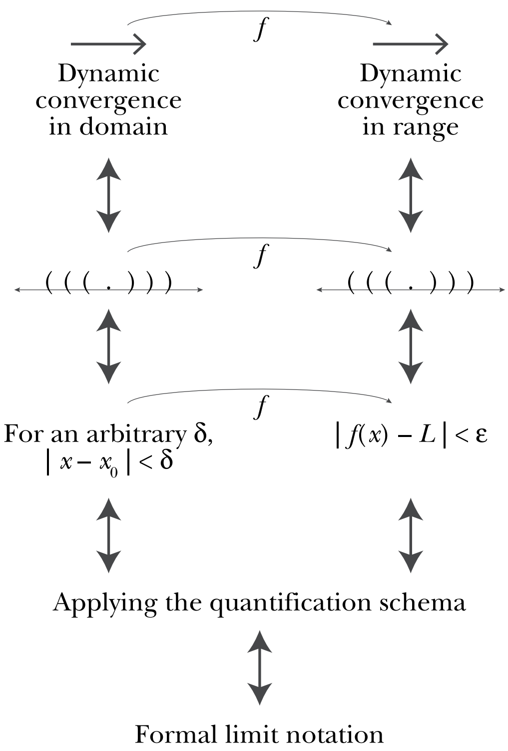


Figure 5. Dynamic participants' formalisation route.

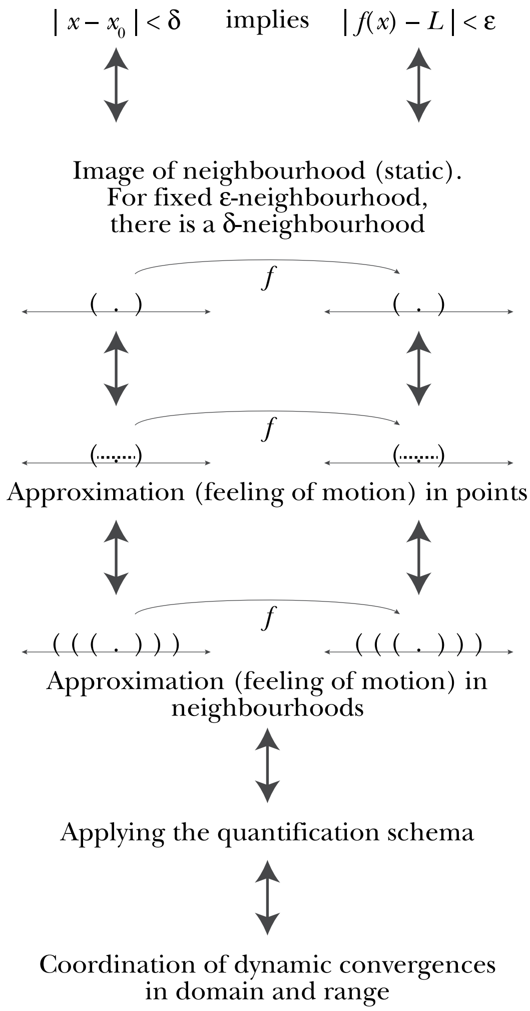


Figure 6. Formal participants' formalisation route.

Both dynamic and formal participants formalised the limit concept by relating dynamic limit notion and formal limit definition. The role of the dynamic limit notion in the formalisation process depended on the different formalisation approaches of the participants. It was thought that the participants may have different kinds of mathematical minds. As cited in Tall (1991), Poincaré (1913) has emphasised that there are two different kinds of mathematical minds. Therefore, it could be the topic of a future study

investigating the relation between students' mathematical minds and their formalisation processes of the limit concept.

According to Cottrill et al. (1996), the schema of two coordinated processes (as $x \to a$, $f(x) \to L$) is reconstructed in terms of the intervals and inequalities in order to obtain a process described as $0 < |x - a| < \delta$ implies $f(x) - L < \varepsilon$. Cottrill et al. have concluded that the coordination of dynamic processes is reconstructed in terms of neighbourhoods, therefore, it is reconstructed in terms of the intervals and inequalities in cognitive development of limit concept. It was seen that this formalisation process, which Cottrill et al. claim, was partially compatible with the conclusion drawn from this study in terms of a dynamic student's formalisation route. However, a formal student's formalisation process of limit concept revealed in this study is not compatible with the formalisation process that Cottrill and his colleagues developed.

As a result, contrary to the claim that the dynamic limit notion prevents the students from formalising the limit notion (e.g., Szydlik, 2000; Williams, 1991), dynamic or formal, none of the limit notions is an obstacle for the students to understand them. The point, which needs to be clarified in the learning process of the limit concept, is about how the students relate the quantifications in formal limit definition to the feeling of motion in the dynamic limit notion, because there were not any students who were able to relate them. Although it has seen that this point is related to the students' understanding of the quantifications, there is a lack of study about this aforementioned need.

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